Original Article

Modeling hedge fund leverage via power utility with subsistence

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ABSTRACT **We use a power utility function with subsistence to model leverage. We prove that as the value of the subsistence level grows the allocation in the risky asset increases. The implication of this result is that if a hedge fund uses leverage, or a high water mark, it tends to have a more aggressive investment strategy. In addition, we prove that the total risk of a portfolio held by an investor with preferences described by a power utility with subsistence is a weighted sum of the covariances between the portfolio's return and higher order powers of that return, shifted by the subsistence level.**
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INTRODUCTION

Loss aversion is a well-known phenomenon in behavioral finance, representing an investor's greater aversion to losses than preference for corresponding gains. When facing a loss, a loss averse investor may switch from a risk-averse strategy to a risk-seeking strategy in order to avoid the loss. Hedge fund management can serve as an illustration for such behavior. In extreme situations when the hedge fund might incur a substantial loss, the risk preferences of the managers may change. The result can be that a hedge fund manager may increase leverage and shift allocation to riskier instruments, as taking on more risk could result in a better return and increase the probability that the hedge fund survives.

The recent financial crisis 2007–2008 has many examples illustrating the above phenomenon. Gregoriou and Lhabitant (2011) argue that one of the causes for the crisis is the increased use of leverage when hedge fund returns were disappointing. In addition, they report that the number of hedge funds that died in 2008 was close to 30 per cent of the overall universe of hedge funds. In their analysis of the financial crisis, leverage is identified to be one of the main causes. Different hedge fund styles use different levels of borrowing. For example, a relative-value style fund typically uses a leverage level of 5–10 times, whereas fixed-income arbitrage or statistical arbitrage styles use leverage levels of 10–20 times. Gregoriou and Lhabitant conclude that 'greed, combined with leverage, misaligned incentives, and complacency, caused the 2008 crisis'. Xu *et al* (2011) report similar results, indicating that the large-scale redemptions and closure of nearly 1500 hedge funds during the 2008 crisis represented a downturn not seen since the expansion of hedge funds in the 1990s. Wiethuechter (2010) studied the contribution of hedge funds to systemic instability of the financial markets with a focus on the 2008 crisis. He indicates that in the last decade there has been an increase in loan-based, or leveraged, investing, and in 2007, the ratio of hedge-fund assets to market positions was 1–5. Similar results can be found in Lo (2008). There are many examples of large hedge fund blow-ups, with high-profile instances including: Long-Term Capital Management (see Halstead *et al*, 2005); Amaranth (see Till, 2008; Chincarini, 2007; Martin, 2007; and Boyd *et al*, 2011); Madoff (see Schneeweis and Szado, 2010; and Bernard and Boyle, 2009). A common theme in these instances is increasing leverage to generate returns as losses start to accumulate.

We model portfolio allocation under such preferences using the notion of a subsistence level (see, for example, Campbell *et al*, 1997). The idea of adding a subsistence level to the usual power utility function has also been called introducing habit formation to the power utility. In a dynamic setting, Constantinides (1990) and Sundaresan (1989) argue for the importance of habit formation and its effect on marginal utility of consumption. Habit formation can arise either as a ratio model as in Abel (1990) or as a difference model as in Sundaresan (1989), Constantinides (1990) and Campbell and Cochrane (1999). In the difference model, the agent's risk aversion varies with the level of consumption relative to habit. We take the view that this property of the utility function makes it appropriate for modeling the behavior of hedge funds that we discuss above with respect to leverage. When the subsistence level is positive the hedge fund is leveraged. Moreover, when the subsistence level is negative, the hedge fund does not invest all available capital. See Chapter 8.4 of Campbell *et al* (1997) for further properties of ratio and difference models and for more on general utility functions with habit formation.

Hedge funds often include a high-water mark in assessing fees and in computing bonuses. This type of benchmark can also be interpreted as a subsistence level or habit formation. A hedge fund manager who operates under a high-water mark rule may invest and trade differently relative to a manager who does not have such a rule because in the former case, the manager only receives a bonus after the high-water mark return is obtained.

The hyperbolic absolute risk aversion (HARA) class of utilities implicitly models a notion of subsistence. In particular, under a positive risk-aversion parameter, the HARA utility places a lower bound on wealth, in similar fashion to the power utility with a subsistence level. Huang and

Litzenberger (1988) refer to the class of HARA utilities we consider as extended power utility functions. The well-known constant absolute risk aversion (CARA) class of utilities arises in the limit as this risk-aversion parameter grows large, and the constant relative risk aversion (CRRA) class also arises as a limiting case. We establish results for the power utility function with subsistence.

For an investor with such preferences, we show that his total portfolio risk measure includes the portfolio variance as well as a weighted sum of the covariances between the portfolio's return and higher order powers of that return, shifted by the subsistence level. In addition, we prove that as the value of the subsistence level grows the allocation in the risky asset increases. The implication of this result is that if a hedge fund uses leverage or a high-water mark, it tends to have a more aggressive investment strategy. We also prove that as the risk-aversion parameter grows for the power utility with subsistence, the magnitude of the allocation in the risky asset decreases. The interpretation of this result is that for a hedge fund with fixed leverage, the strategy becomes more conservative as risk aversion increases.

Our approach to characterizing the total portfolio risk is rooted in a Taylor series expansion. That said, we show that under the power utility function with subsistence, these terms can be expressed as a covariance between the portfolio return and higher order powers of the return shifted by the subsistence level.

UTILITY FUNCTIONS WITH SUBSISTENCE

Consider an investor with initial wealth W_0, who has an objective to maximize his final wealth, $\tilde{W}$, that is, the cumulative portfolio return. Let r_f be the risk-free rate of return and $R_f = 1 + r_f$ denote the total risk-free return. Assume for the moment that there is one risky asset with end of period random return $\tilde{r}$. Let h denote the weight the investor places on the risky asset, and $1-h$ be the weight on the risk-free asset. Under the assumption that $W_0 = 1$, the end of period investor's wealth is then $\tilde{W} = 1 + r_f + h(\tilde{r} - r_f)$. (We subsequently extend the discussion to multiple risky assets.)

Next, we assume that the investor has a power utility function of final wealth with a subsistence level:

$$u(\tilde{W}) = \frac{(\tilde{W} + A)^{1-\gamma}}{1-\gamma} \qquad (1)$$

where A is a constant and $\gamma > 0$, $\gamma \neq 1$, measures the investor's risk aversion. In utility (1), given our assumption that $W_0 = 1$, the parameter A can be interpreted as a percentage, and we refer to A as a subsistence level. If $A = 0$, then $u(\tilde{W})$ in (1) reduces to a narrow power utility function. When $A>0$, the investor is leveraged, that is, he borrowed money in addition to his initial wealth. When $A<0$, the investor is not using all of his wealth, or he might have lent some of it before considering investing in the risky asset. That said, when interpreting the shift as subsistence, utility (1) seems to cleanly separate the notions of risk aversion and subsistence in its parameters. This intuition is supported in the formal results that follow, establishing the sensitivity of the optimal allocation h to the risky asset with respect to parameters A and γ.

Note that the utility function, $u(\cdot)$, in equation (1) only makes sense if the random variable $\tilde{W} + A$ is nonnegative. This is not a practical restriction in the sense that the worst possible return should result in losing all invested money, that is, should result in an argument of zero in the function on the right-hand side of equation (1). Of course, this restriction is not captured mathematically by all probability distributions that one might use (for example, a normal distribution governing the return of the risky asset). For this reason, we explicitly require this property in the formal results that follow. We further assume throughout that the expectation $Eu(\tilde{W})$ exists for the $\tilde{W}$ resulting from any investment.

Proposition 1: Let $\gamma>0$, $\gamma\neq 1$, and let $u(\cdot)$ be the power utility function with subsistence defined in equation (1). Let $\tilde{W}=1+r_f+h(\tilde{r}-r_f)$, assume $H=\{h:\tilde{W}+A\geqslant 0$, with probability one$\}\neq 0$, and assume that $\tilde{r}$ is nondegenerate. Let h^* solve $\max_{h\epsilon H} Eu(\tilde{W})$, and assume h^* is an interior point of H. Then, $dh^*/dA>0$.

Proof See Appendix. □

Proposition 2: Assume that the hypotheses of Proposition 1 hold and that $h^*\neq 0$. For the power utility function with subsistence defined in equation (1) $d|h^*|/d\gamma<0$.

Proof See Appendix. □

Propositions 1 and 2 establish results for how the allocation in the risky asset changes with changes in the value of the subsistence parameter, A, and the exponent parameter, γ, for the power utility function with subsistence (1) that are consistent with intuition. In the proof of Proposition 1, we show that as A grows, our risk aversion decreases and our allocation to the risky asset grows. On the other hand, as γ grows our risk aversion grows and Proposition 2 shows that the magnitude of our allocation to the risky asset drops.

For the power utility under subsistence, we illustrate the behavior of h^* and $dh^*/d\gamma$ for different values of the risk-free rate r_f, assuming that the return of the risky asset follows a normal distribution with mean, $\mu=10$ per cent, and standard deviation, $\sigma=20$ per cent, and with a subsistence level of $A=0$. Figure 1 shows how the weight of the optimal allocation changes with the risk-free rate for two different values of γ.

The allocation in the risky asset shrinks as the risk-free rate grows and becomes zero when the risk-free rate reaches the expected return of the risky asset, that is, when $r_f=E[\tilde{r}]=10$ per cent. Once the risk-free rate takes a value larger

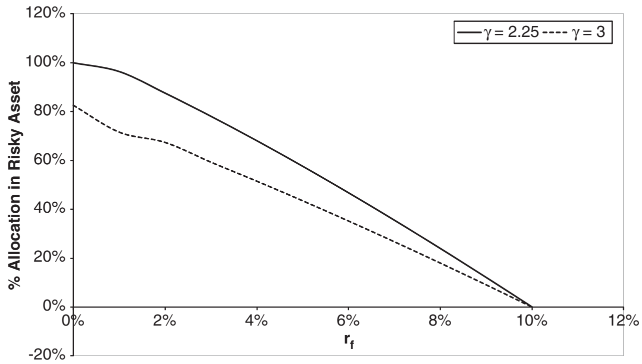

Figure 1: Optimal allocation in the risky asset, h^*, as a function of the risk-free rate. We assume that the return of the risky asset follows a normal distribution with mean, $\mu=10$ per cent, and standard deviation, $\sigma=20$ per cent. The subsistence level is $A=0$.

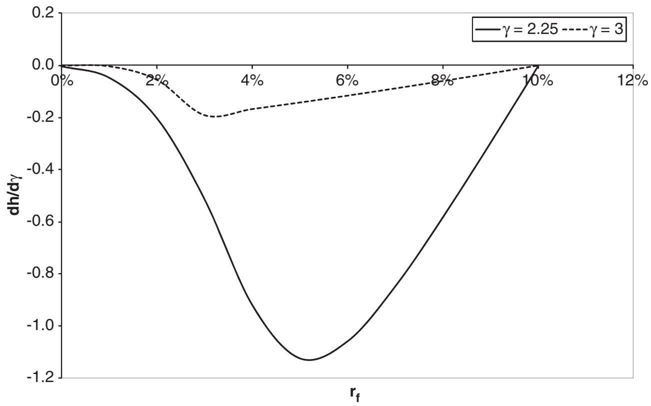

Figure 2: The derivative of the optimal allocation in the risky asset with respect to the risk-aversion parameter, $dh^*/d\gamma$, as a function of the risk-free rate. We assume that the return of the risky asset follows a normal distribution with mean, $\mu = 10$ per cent, and standard deviation, $\sigma = 20$ per cent. The subsistence level is $A = 0$.

than $E[\tilde{r}]$, any positive allocation in the risky asset is no longer optimal. Figure 2 shows how $dh^*/d\gamma$ varies with the risk-free rate, with the same two values of γ.

For all values of $r_f \leq E[\tilde{r}]$, $dh^*/d\gamma < 0$, and the derivative reaches zero when $r_f = E[\tilde{r}]$.

TOTAL PORTFOLIO RISK

We extend the notation of the previous section to capture an asset allocation problem with N risky assets indexed by $i = 1, \ldots, N$. These assets have random returns $\tilde{r} = (r_i, \ldots, r_N)$ and random total returns $\tilde{R} = (\tilde{R}_1, \ldots, \tilde{R}_N) \geqslant 0$, where $\tilde{R}_i = \tilde{r}_i + 1$, $i = 1, \ldots, N$. As in the previous section, the risk-free rate of return is denoted r_f and the total risk-free return is $R_f = 1 + r_f$. We denote allocations to the risky assets by the vector $h = (h, \ldots, h_N)$, and the allocation to the risk-free asset is then $1 - \Sigma_{i=1}^{N} h_i$. Again assuming $W_0 = 1$, we have that terminal wealth is $\tilde{W} = 1 + r_f + \Sigma_{i=1}^{N} h_i(\tilde{r}_i - r_f)$.The total return of the portfolio is defined via $R_p = \tilde{W}/W_0$ and hence is synonymous with terminal wealth, when assuming unit initial wealth. In the same manner as above, we define the portfolio's return as $r_p = R_p - 1$.

We can then state our asset allocation problem as:

$$\max_h Eu(\tilde{W}) \tag{2}$$

Here, the utility function $u(\cdot)$ can be that in equation (1) or could be some other utility. Consider the asset allocation model (2) and assume that the utility function u is concave and twice continuously differentiable. In this case, the total portfolio risk is defined as:

$$\Omega_p = \mathrm{Cov}\left(r_p, \frac{u'(1 + r_p)}{Eu''(1 + r_p)} \right) \tag{3}$$

We now motivate how this arises. The first-order necessary and sufficient conditions that an optimal solution of (2) satisfies are given by:

$$E\left[u'(1 + r_p)(\tilde{r}_i - r_f)\right] = 0, \; i = 1, \ldots, N. \tag{4}$$

Multiplying these respective terms by h_i^* and summing yields:

$$E[u'(1+r_p)(r_p-r_f)]=0. \tag{5}$$

Equation (5), coupled with the definition of covariance yields:

$$\begin{aligned} E[r_p-r_f] &= \frac{-1}{Eu'(1+r_p)}\mathrm{Cov}(r_p-r_f, u'(1+r_p)) \\ &= \underbrace{\frac{-Eu''(1+r_p)}{Eu'(1+r_p)}}_{\theta} \times \underbrace{\mathrm{Cov}\left(r_p, \frac{u'(1+r_p)}{Eu''(1+r_p)}\right)}_{\Omega_p} \end{aligned} \tag{6}$$

The Arrow–Pratt measure of absolute risk aversion is defined by $-u''(\cdot)/u'(\cdot)$. Of course, this is a function rather than a single value. Huang and Litzenberger (1988) define global absolute risk aversion via $\theta=-Eu''(1+r_p)/Eu'(1+r_p)$, a single value. See Bardsley (1991) for more on this same notion. As a result, equation (6) relates the portfolio's excess return over the risk-free rate to a measure of the investor's risk aversion and Ω_p. This justifies labeling the latter term a measure of risk, termed the total portfolio risk. Under the optimal allocation, $r_p = r_f + \sum_{i=1}^{N} h_i^*(\tilde{r}_i - r_f)$, and we have:

$$\begin{aligned} \Omega_p &= \mathrm{Cov}\left(r_p, \frac{u'(1+r_p)}{Eu''(1+r_p)}\right) \\ &= \sum_{i=1}^{N} h_i^* \underbrace{\mathrm{Cov}\left(\tilde{r}_i, \frac{u'(1+r_p)}{Eu''(1+r_p)}\right)}_{\lambda_i} \end{aligned} \tag{7}$$

where λ_i is called the marginal risk of asset i.

It is well known that for general return distributions (that is, including non-normal distributions), the mean–variance model can be motivated by assuming a quadratic utility; see, for example, Huang and Litzenberger (1988), p. 61, or Levy and Markowitz (1979). We summarize this result, in the context of total portfolio risk, in the following proposition.

Proposition 3: For a quadratic utility function

$$u(\tilde{W}) = \tilde{W} - \frac{b\tilde{W}^2}{2}$$

where $b>0$, the total portfolio risk is given by: $\Omega_p = \mathrm{Var}(r_p)$.

Proof See Appendix. □

The quadratic utility is only sensible when wealth is restricted to the range, on which the utility is increasing, in our case $\tilde{W} < 1/b$. Proposition 3 shows that even if the underlying asset return distribution is not normal, if the investor has a quadratic utility, his measure for total portfolio risk is the portfolio variance. The next proposition shows that for an investor with a power utility function under subsistence, the total portfolio risk is not just the variance.

Proposition 4: Let $\gamma>0, \gamma\neq 1$, and let $u(\cdot)$ be the power utility with subsistence defined in equation (1). For the power utility with subsistence let $R_u = R_p + A > 0$, w.p.1, for all investments. We similarly define $r_u = r_p + A$. Assume that the optimal allocation in model (2) satisfies $R_u < 2$, w.p.1. Then the total portfolio risk, Ω_p, is given by:

$$\begin{aligned} \Omega_p = &\frac{\mathrm{Var}(r_p)}{E\left[R_u^{-(1+\gamma)}\right]} \\ &+ \frac{1}{E\left[R_u^{-(1+\gamma)}\right]} \\ &\times \sum_{n=2}^{\infty} \frac{(-1)^{n+1}\Gamma(\gamma+n)}{n!\,\Gamma(\gamma+1)} \mathrm{Cov}\left[r_p, r_u^n\right] \end{aligned} \tag{8}$$

where $\Gamma(\cdot)$ is the gamma function.

Proof See Appendix. □

Proposition 3 shows that for an arbitrary distribution with finite second moments, the total portfolio risk is represented by the variance of the portfolio returns when the investor has a quadratic utility function. On the other hand, Proposition 4 shows that the portfolio variance is not the only factor contributing to the total portfolio risk when the investor has a power utility with subsistence. From equation (8) it is clear that higher moments from the portfolio return distribution also contribute to the total portfolio risk. More specifically, Ω_p involves a weighted sum of covariance terms between the portfolio's return, r_p, and higher order powers of that return shifted by the level of subsistence. That shift is A for the power utility with subsistence.

CONCLUSION

Variance is the total portfolio risk for an investor who has a quadratic utility function. We show that for an investor who has preferences described by a power utility with subsistence, the total portfolio risk includes the variance and a weighted sum of the covariances between the portfolio's return and higher order powers of that return, shifted by the subsistence level.

We prove that as the value of the subsistence level grows the allocation in the risky asset increases. The implication of this result is that if a hedge fund uses leverage or awards bonuses based on high-water marks, it tends to have a more aggressive investment strategy. Our recommendation to institutional investors, and managers of funds of hedge funds, is to incorporate additional tools in their process of due diligence when considering investing in a hedge fund that uses leverage. In particular, hedge-fund investment strategies such as convertible arbitrage, equity market neutral, global macro and long–short equity, all use leverage in order to enhance returns. The due diligence process should pay special attention to crisis periods, specifically to examine whether the hedge fund's leverage was increased after a significant loss. Such an event should be a red flag for potential future losses.

We also prove that as the risk-aversion parameter of the power utility function under subsistence grows, the allocation in the risky asset decreases in magnitude. The interpretation of this result is that for a hedge fund with a fixed-leverage investment style, the strategy becomes more conservative as this risk-aversion parameter increases.

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APPENDIX

PROOF OF PROPOSITION 1

Under the assumption that h^* is an interior point of H, the optimality condition that specifies h^* is: $E\left[u'(\tilde{W})(\tilde{r}-r_f)\right]=0$. Using this expression to implicitly differentiate h^* with respect to A gives:

$$E\left[u''(\tilde{W})\left(1+(\tilde{r}-r_f)\frac{dh^*}{dA}\right)(\tilde{r}-r_f)\right]=0,$$

and then solving for dh^*/dA yields:

$$\frac{dh^*}{dA}=\frac{E\left[u''(\tilde{W})(\tilde{r}-r_f)\right]}{-E\left[u''(\tilde{W})(\tilde{r}-r_f)^2\right]}.$$

The denominator is positive because u is strictly concave and hence,

$$sign\left(\frac{dh^*}{dA}\right)=sign\left(E\left[u''(\tilde{W})(\tilde{r}-r_f)\right]\right).$$

Absolute risk aversion of a utility $u(\cdot)$ is defined as $R_A(\cdot)=-u''(\cdot)/u'(\cdot)$, and hence for utility function (1), $R_A(\tilde{W})=\gamma/(\tilde{W}+A)$ is a strictly decreasing function of terminal wealth. Restated, the power utility $u(\cdot)$ under subsistence has decreasing absolute risk aversion. We can now complete the proof using the following fact from Huang and Litzenberger (1988), p. 22: If $u(\cdot)$ has decreasing absolute risk aversion then the sign of $E\left[u''(\tilde{W})(\tilde{r}-r_f)\right]$ is positive. As a result of this fact, we can conclude $dh^*/dA>0$.

PROOF OF PROPOSITION 2

We begin as in the proof of Proposition 1, that is, with the optimality condition $E\left[u'(\tilde{W})(\tilde{r}-r_f)\right]=0$. Using this expression to implicitly differentiate h^* with respect to γ yields:

$$E\left[u'(\tilde{W})\left(-ln(\tilde{W}+A)-\frac{\gamma}{\tilde{W}+A}(\tilde{r}-r_f)\frac{dh^*}{d\gamma}\right)(\tilde{r}-r_f)\right]=0.$$

Solving for $dh^*/d\gamma$ gives:

$$\frac{dh^*}{d\gamma}=\frac{E\left[u'(\tilde{W})ln(\tilde{W}+A)(\tilde{r}-r_f)\right]}{-E\left[\gamma\frac{u'(\tilde{W})}{\tilde{W}+A}(\tilde{r}-r_f)^2\right]}. \quad (9)$$

Given that the utility function is increasing and $\tilde{W}+A\geqslant 0$, the denominator is negative.

Therefore:

$$sign\left(\frac{dh^*}{d\gamma}\right) = -sign\left(E\left[u'\left(\tilde{W}\right)ln\left(\tilde{W}+A\right)\left(\tilde{r}-r_f\right)\right]\right).$$

Assume for the moment that $h^*>0$. Then,

$$\ln\left(\tilde{W}+A\right) > \ln(1+r_f+A),\ if\ \tilde{r} > r_f \quad (10)$$

$$\ln\left(\tilde{W}+A\right) < \ln(1+r_f+A),\ if\ \tilde{r} < r_f. \quad (11)$$

Multiplying by $u'\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)$ and taking expectations yields:

$$E[u'(\tilde{W})\ln(\tilde{W}+A)(\tilde{r}-r_f)] > E[u'(\tilde{W})(\tilde{r}-r_f)]\ln(1+r_f+A)=0,$$

by the optimality condition $E\left[u'\left(\tilde{W}\right)\left(\tilde{r}-r_f\right)\right]=0$. Repeating the argument for $h^*<0$, the inequalities in (10) and (11) reverse and we obtain

$$\left[u'\left(\tilde{W}\right)\ln\left(\tilde{W}+A\right)\left(\tilde{r}-r_f\right)\right] < 0.$$

This establishes that $dh^*/d\gamma < 0$ when $h^* > 0$ and $dh^*/d\gamma > 0$ when $h^* < 0$, and completes the proof.

PROOF OF PROPOSITION 3

Under the quadratic utility function

$$\frac{u'\left(1+r_p\right)}{E\left[u''\left(1+r_p\right)\right]} = -\frac{1}{b}+1+r_p.$$

Substitution of this result into equation (3), and using the fact that covariance is invariant under constant translations, we have that

$$\Omega_p = \mathrm{Cov}\left(r_p,\ -\frac{1}{b}+1+r_p\right) = \mathrm{Cov}\left(r_p,\ r_p\right) = \mathrm{Var}\left(r_p\right).$$

PROOF OF PROPOSITION 4

Under the power utility function with subsistence,

$$u'\left(1+r_p\right) = \left(1+r_p+A\right)^{-\gamma} \quad (12)$$

$$u''\left(1+r_p\right) = -\gamma\left(1+r_p+A\right)^{-(1+\gamma)}. \quad (13)$$

The binomial series for (12) converges provided $|r_p+A|<1$, which is ensured by hypothesis.

Forming that series we have:

$$\left(1+r_p+A\right)^{-\gamma} = \sum_{n=0}^{\infty}\frac{(-1)^n\Gamma(\gamma+n)}{n!\Gamma(\gamma)}\left(r_p+A\right)^n,$$

and hence using (13):

$$\frac{u'\left(1+r_p\right)}{E\left[u''\left(1+r_p\right)\right]} = \frac{1}{E\left[\left(R_p+A\right)^{-(1+\gamma)}\right]} \times \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\left(r_p+A\right)^n.$$

Substitution of this expression into (3) yields:

$$\begin{aligned}\Omega_p &= \frac{1}{E\left[\left(R_p+A\right)^{-(1+\gamma)}\right]} \times \mathrm{Cov}\left[r_p, \sum_{n=0}^{\infty}\frac{(-1)^{n+1}\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\left(r_p+A\right)^n\right] \\ &= \frac{1}{E\left[\left(R_p+A\right)^{-(1+\gamma)}\right]}\left\{\mathrm{Var}\left(r_p\right) + \mathrm{Cov}\left[r_p, \sum_{n=2}^{\infty}\frac{(-1)^{n+1}\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\left(r_p+A\right)^n\right]\right\} \\ &= \frac{1}{E\left[\left(R_p+A\right)^{-(1+\gamma)}\right]}\left\{\mathrm{Var}\left(r_p\right) + \sum_{n=2}^{\infty}\frac{(-1)^{n+1}\Gamma(\gamma+n)}{n!\Gamma(\gamma+1)}\mathrm{Cov}\left[r_p,\ \left(r_p+A\right)^n\right]\right\},\end{aligned}$$

which completes the proof.